Casimir operators of the exceptional group F_4: the chain B_4 contained in/implied by F_4 contained in/implied by D_{13}

1994 J. Phys. A: Math. Gen. 27 3847

(http://iopscience.iop.org/0305-4470/27/11/033)

J. Phys. A: Math. Gen. 27 (1994) 3847–3856. Printed in the UK

Casimir operators of the exceptional group F_4: the chain $B_4 \subset F_4 \subset D_{13}$

Adam M Bincer
Department of Physics, University of Wisconsin–Madison, Madison, WI 53706, USA

Received 4 January 1994, in final form 22 March 1994

Abstract. Expressions are given for the Casimir operators of the exceptional group F_4 in a concise form similar to that used for the classical groups. The chain $B_4 \subset F_4 \subset D_{13}$ is used to label the generators of F_4 in terms of the adjoint and spinor representations of B_4 and to express the 26-dimensional representation of F_4 in terms of the defining representation of D_{13}. Casimir operators of any degree are obtained and it is shown that a basis consists of the operators of degree 2, 6, 8 and 12.

1. Introduction

Although a general formula exists for the quadratic Casimir operator for any group this is not the case for operators of higher degree. Efficient expressions have been developed over the years for all the Casimir operators of the classical groups, but not for the exceptional groups. Berdjis [1] gives the desired Casimir operators implicitly. Until recently, explicit results were available only for G_2. The degree-6 Casimir of G_2 was given in the work of Hughes and Van der Jeugt [2] by an expression involving 29 terms and in the work by Bincer and Riesselmann [3] by an expression involving 23 terms. These results were obtained using computers and leave something to be desired.

Quite recently I have developed a different approach and obtained for G_2 results very much alike to those for the classical groups [4]. Moreover, it would seem that the same approach should work for the other exceptional groups. In the present work I address the group F_4 and leave $E_{6,7,8}$ for a future paper.

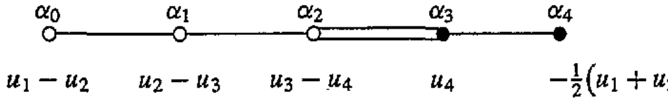
This paper is organized as follows. In the next section, after explaining the use of the chain $B_4 \subset F_4 \subset D_{13}$, I obtain concise expressions for the Casimir operators of F_4. These require the knowledge of the generators of D_{13} projected into F_4. To obtain this projection I describe in the next section the 26-dimensional representation of F_4 and then obtain in the following section the desired projection. In the conclusion I discuss the quadratic Casimir operator of F_4 and demonstrate that the independent Casimir operators are of degree 2, 6, 8 and 12 (corresponding to the exponents of F_4 being 1, 5, 7 and 11).

2. The Casimir operators of D_{13} and F_4

My approach makes use of the chain $B_4 \subset F_4 \subset D_{13}$. The subgroup B_4 of F_4 is used to label the generators of F_4. F_4 is embedded in D_{13} because the smallest-dimensional representation of F_4 is 26-dimensional and orthogonal and D_{13} is the orthogonal group in 26 dimensions.

0305-4470/94/113847+10$19.50

I denote the 36 generators of B_4 as $B_\alpha^\beta = -B_{\bar\beta}^{\bar\alpha}$, with indices ranging from -4 to $+4$, zero *included*, $\bar\alpha \equiv -\alpha$. The Hermitian property is expressed in this basis as $B_\alpha^{\beta\dagger} = B_\beta^\alpha$. I denote the generators of F_4 as B_α^β and S^{pqrs}, corresponding to the decomposition of the **52** (the adjoint) of F_4 into the **36** and **16** of B_4, where the **36** is the adjoint, i.e. the B_α^β, and the **16** is the spinor $S^{pqrs} = (S^{\bar p\bar q\bar r\bar s})^\dagger$, $p, q, r, s = \pm$. The $B_4 \subset F_4$ relation is exhibited in the extended Dynkin diagram



with B_4 obtained by omitting α_4 and F_4 obtained by omitting α_0. That is to say, the α_i, $1 \leqslant i \leqslant 4$, are the simple roots of F_4, while the α_j, $0 \leqslant j \leqslant 3$ are the simple roots of B_4. The information encoded in the Dynkin digram is made explicit by setting $\alpha_0 = u_1 - u_2, \alpha_1 = u_2 - u_3, \alpha_2 = u_3 - u_4, \alpha_3 = u_4, \alpha_4 = -\frac{1}{2}(u_1 + u_2 + u_3 + u_4)$, where the u_i are orthogonal unit vectors.

I denote the generators of D_{13} as $D_a^b = -D_{\bar b}^{\bar a}, (D_a^b)^\dagger = D_b^a$, zero *excluded*. The commutation relations of D_{13} in this basis are

$$\left[D_a^b, D_c^d\right] = \delta_c^b D_a^d - \delta_a^d D_c^b + \delta_{\bar b}^d D_c^{\bar a} - \delta_c^{\bar a} D_{\bar b}^d . \tag{1}$$

It follows from (1) that

$$\left[D_a^b, (D^k)_c^d\right] = \delta_c^b (D^k)_a^d - \delta_a^d (D^k)_c^b + \delta_{\bar b}^d (D^k)_c^{\bar a} - \delta_c^{\bar a} (D^k)_{\bar b}^d \tag{2}$$

where I define the kth power, $k \geqslant 1$, by

$$(D^k)_a^b = (D^{k-1})_a^c D_c^b = D_a^c (D^{k-1})_c^b \qquad (D^0)_a^b = \delta_a^b \tag{3}$$

(summation convention understood). It now follows that if I define

$$C_k(D_{13}) = (D^k)_a^a \tag{4}$$

then these C_k commute with the generators of D_{13} and so are Casimir operators of D_{13} of degree k. Equation (14) provides an elegant expression for the Casimir operators of D_{13} and is an example of the type of expressions valid for all the classical groups. All this is well known and goes back to Perelomov and Popov [5]. I remark that the 13 independent Casimirs of this type are of degree $k = 2s$, $1 \leqslant s \leqslant 13$. This is because it follows from the antisymmetry property $D_a^b = -D_{\bar b}^{\bar a}$ that the Casimirs for $k =$ odd can be expressed in terms of those for $k =$ even, and it follows from the Cayley–Hamilton theorem that Casimirs of degree $k > 26$ can be expressed in terms of those for $k \leqslant 26$. I note further that the Casimir operator of degree 26 can be expressed in terms of the square of a Casimir of degree 13 (which is not of the form given by (4)) and so the integrity basis for the Casimirs contains those of degree $k = 2s$, $1 \leqslant s \leqslant 12$, and $k = 13$, which agrees with the fact that the degrees k of the Casimirs in the basis should be equal to the exponents of D_{13} plus one.

We next observe that, under the restriction of D_{13} to F_4, the adjoint representation of D_{13} decomposes thus

$$\mathbf{325} = \mathbf{52} + \mathbf{273} \tag{5}$$

where the **325** refers to the adjoint of D_{13} and the **52** to the adjoint of F_4. Thus we can express the generators D_a^b of D_{13} in terms of the generators of F_4 and the components of the **273**-plet. We now obtain the Casimir operators of F_4 by observing that they are given by (4) in which the D_a^b are replaced by their projections into F_4, i.e.

$$C_k(F_4) = (\tilde D^k)_a^a \tag{6}$$

where

$$\tilde{D}_a^b = D_a^b|_{273=0} \,. \tag{7}$$

I mean by (7) that the projected $\tilde{D}_a^b$ are given by expressing the D_a^b in terms of the generators of F_4 and members of the **273**-plet and then setting the contribution of the **273**-plet equal to zero.

3. The 26-dimensional representation of F_4

To obtain the projected $\tilde{D}$ I need to obtain first explicit formulae for the 26-dimensional representation of F_4.

The generators D_a^b of D_{13} are given in the defining 26-dimensional representation as the following 26×26 matrices:

$$D_a^b = I_{ab} - I_{\overline{ba}} \tag{8}$$

where I_{ab} is the 26×26 matrix with matrix elements

$$(I_{ab})_{jk} = \delta_{aj}\delta_{bk} \tag{9}$$

with the labels j, k taking on the same values as a, b: $-13 \leqslant j, k \leqslant 13$, zero excluded.

The Cartan generators of F_4 are given in the 26-dimensional representation by the 26×26 matrices as follows:

$$h_1 = D_5^5 + D_6^6 - D_7^7 + D_8^8 - D_9^9 - D_{10}^{10} \tag{10}$$

$$h_2 = D_3^3 + D_4^4 - D_5^5 - D_6^6 + D_{10}^{10} - D_{11}^{11} \tag{11}$$

$$h_3 = \tfrac{1}{2}\left(D_2^2 - 2D_3^3 - D_4^4 + D_6^6 - D_8^8 + D_9^9 - D_{10}^{10} + D_{11}^{11} - D_{12}^{12}\right) \tag{12}$$

$$h_4 = \tfrac{1}{2}\left(-2D_2^2 + D_3^3 - D_4^4 + D_5^5 - D_6^6 + D_7^7 - D_9^9 + D_{12}^{12} - D_{13}^{13}\right). \tag{13}$$

These are precisely the same expressions as were obtained by Patera [6] and Ekins and Cornwell [7] if I relabel their rows and columns thus: their 1 $\to$ mine $-$ 13, their 2 $\to$ mine $-$ 12, ..., their 13 $\to$ mine $-$ 1, their 14 $\to$ mine + 1, ..., their 26 $\to$ mine + 13.

Given these explicit matrices for the Cartan generators h_i, the associated generators e_i and $f_i = e_i^\dagger$ in the Chevalley basis are found from the equations [7]

$$\left[e_j, h_k\right] = A_{kj}e_j \qquad \left[f_j, e_k\right] = \delta_{jk}h_k \,. \tag{14}$$

The summation convention does not apply to (14) and A is the Cartan matrix of F_4,

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{15}$$

A solution of (14) for the simple generators e_i is as follows:

$$e_1 = D_7^5 + D_9^6 + D_{10}^8 \tag{16}$$

$$e_2 = D_5^3 + D_6^4 + D_{11}^{10} \tag{17}$$

$$e_3 = 2^{-\frac{1}{2}}\left(D_3^{\bar{1}} + D_3^1 + D_4^2 + D_8^6 + D_{10}^9 + D_{12}^{11}\right) \tag{18}$$

$$e_4 = 2^{-\frac{1}{2}}\left(zD_2^{\bar{1}} + z^*D_2^1 + D_4^3 + D_6^5 + D_9^7 + D_{13}^{12}\right) \tag{19}$$

where

$$z \equiv e^{i\pi/3} \,. \tag{20}$$

Except for the renumbering of rows and columns and a different choice of phases, my expressions for e_1 and e_2 are precisely the same as those given by Patera [6] and Ekins and Cornwell [7]. However, my expressions for e_3 and e_4 differ from the corresponding expressions of those authors. It would seem that they resolved some of the arbitrariness in the solution by demanding that it be real; I require that it display the antisymmetry across the antidiagonal corresponding to the fact that we have an orthogonal representation.

In accordance with my labelling of generators of F_4 in the B_4 basis in terms of the adjoint and the spinor of B_4 I have that the above simple generators e_i should be labelled as follows:

$$\begin{aligned} &\alpha_1 = u_2 - u_3 \to e_1 = B_2^3 \qquad \alpha_2 = u_3 - u_4 \to e_2 = B_3^4 \\ &\alpha_3 = u_4 \to e_3 = B_4^0 \qquad \alpha_4 = -\tfrac{1}{2}(u_1 + u_2 + u_3 + u_4) \to e_4 = S^{++++} . \end{aligned} \tag{21}$$

Next I form commutators of the simple generators and obtain level one generators:

$$\begin{aligned} \alpha_1 + \alpha_2 &= u_2 - u_4 \to B_2^4 = [B_2^3, B_3^4] = D_7^3 + D_9^4 - D_{11}^8 \\ \alpha_2 + \alpha_3 &= u_3 \to B_3^0 = [B_3^4, B_4^0] = 2^{-1/2}(D_5^{\bar{1}} + D_5^1 + D_6^2 - D_8^4 + D_{11}^9 - D_{12}^{10}) \\ \alpha_3 + \alpha_4 &= -\tfrac{1}{2}(u_1 + u_2 + u_3 - u_4) \to S^{+++-} = [B_4^0, S^{++++}]\sqrt{2} \\ &= -2^{-1/2}(z^* D_4^{\bar{1}} + z D_4^1 + D_3^{\bar{2}} - D_8^5 - D_{10}^7 + D_{13}^{11}) \end{aligned} \tag{22}$$

level two generators:

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= u_2 \to B_2^0 = [B_2^3, B_3^0] = 2^{-1/2}(D_7^{\bar{1}} + D_7^1 + D_9^2 - D_{10}^4 - D_{11}^6 + D_{12}^8) \\ \alpha_2 + \alpha_3 + \alpha_4 &= -\tfrac{1}{2}(u_1 + u_2 - u_3 + u_4) \to S^{++-+} \\ &= [B_3^0, S^{++++}]\sqrt{2} = -2^{-1/2}(z^* D_6^{\bar{1}} + z D_6^1 + D_5^{\bar{2}} + D_8^3 - D_{11}^7 - D_{13}^{10}) \\ \alpha_2 + 2\alpha_3 &= u_3 + u_4 \to B_3^{\bar{4}} = [B_4^0, B_3^0] = D_5^{\bar{3}} + D_8^2 + D_{12}^9 \end{aligned} \tag{23}$$

level three generators:

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= -\tfrac{1}{2}(u_1 - u_2 + u_3 + u_4) \to S^{+-++} = [B_2^0, S^{++++}]\sqrt{2} \\ &= -2^{-1/2}(z^* D_9^{\bar{1}} + z D_9^1 + D_7^{\bar{2}} + D_{10}^3 + D_{11}^5 + D_{13}^8) \\ \alpha_1 + \alpha_2 + 2\alpha_3 &= u_2 + u_4 \to B_2^{\bar{4}} = [B_4^0, B_2^0] = D_7^{\bar{3}} - D_{12}^6 + D_{10}^2 \\ \alpha_2 + 2\alpha_3 + \alpha_4 &= -\tfrac{1}{2}(u_1 + u_2 - u_3 - u_4) \to S^{++--} = [B_3^0, S^{+++-}]\sqrt{2} \\ &= -2^{-1/2}(z D_8^{\bar{1}} + z^* D_8^1 - D_6^{\bar{3}} - D_5^{\bar{4}} + D_{12}^7 - D_{13}^9) \end{aligned} \tag{24}$$

level four generators

$$\begin{aligned} \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 &= -\tfrac{1}{2}(u_1 - u_2 + u_3 - u_4) \to S^{+-+-} = [B_2^0, S^{+++-}]\sqrt{2} \\ &= -2^{-1/2}(z D_{10}^{\bar{1}} + z^* D_{10}^1 - D_7^{\bar{4}} - D_9^{\bar{3}} - D_{12}^5 + D_{13}^6) \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 &= u_2 + u_3 \to B_2^{\bar{3}} = [B_3^{\bar{4}}, B_2^4] = D_7^{\bar{5}} + D_{12}^4 + D_{11}^2 \\ \alpha_2 + 2\alpha_3 + 2\alpha_4 &= -u_1 - u_2 \to B_{\bar{1}}^2 = [S^{++-+}, S^{+++-}] = D_4^{\bar{6}} + D_8^{\bar{2}} + D_{13}^7 \end{aligned} \tag{25}$$

level five generators:

$$\begin{aligned} \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 &= -u_1 - u_3 \to B_{\bar{1}}^3 = [B_{\bar{1}}^2, B_2^3] = D_9^{\bar{4}} - D_{10}^{\bar{2}} + D_{13}^5 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 &= -\tfrac{1}{2}(u_1 - u_2 - u_3 + u_4) \to S^{+--+} = [B_3^0, S^{+-++}]\sqrt{2} \\ &= 2^{-1/2}(z D_{11}^{\bar{1}} + z^* D_{11}^1 - D_7^{\bar{6}} - D_9^{\bar{5}} + D_{12}^3 - D_{13}^4) \end{aligned} \tag{26}$$

level six generators:

$$\begin{aligned}
&\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 = -u_1 - u_4 \to B_{\bar{1}}^{4} = \left[B_{\bar{1}}^{2}, B_{2}^{4}\right] = D_{6}^{\bar{9}} + D_{11}^{\bar{2}} + D_{13}^{3} \\
&\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 = -\tfrac{1}{2}\left(u_1 - u_2 - u_3 - u_4\right) \to S^{+---} = -\left[B_4^0, S^{+--+}\right]\sqrt{2} \\
&\qquad = 2^{1/2}\left(z^* D_{12}^{\bar{1}} + z D_{12}^{1} + D_{7}^{\bar{8}} + D_{10}^{\bar{5}} - D_{11}^{\bar{3}} - D_{13}^{2}\right)
\end{aligned} \tag{27}$$

one level seven generator:

$$\begin{aligned}
&\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = -u_1 \to B_{\bar{1}}^{0} = -\left[S^{+---}, S^{++++}\right]\sqrt{2} \\
&\qquad = 2^{1/2}\left(D_{13}^{\bar{1}} + D_{13}^{1} + D_{9}^{\bar{8}} + D_{10}^{\bar{6}} - D_{11}^{\bar{4}} - D_{12}^{\bar{2}}\right)
\end{aligned} \tag{28}$$

one level eight generator:

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 = -u_1 + u_4 \to B_4^1 = \left[B_{\bar{1}}^0, B_4^0\right] = -D_{10}^{\bar{8}} + D_{12}^{\bar{4}} - D_{13}^{\bar{3}} \tag{29}$$

one level nine generator:

$$\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = -u_1 + u_3 \to B_3^1 = \left[B_{\bar{1}}^0, B_3^0\right] = -D_{11}^{\bar{8}} + D_{12}^{\bar{6}} - D_{13}^{\bar{5}} \tag{30}$$

and one level ten generator:

$$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = -u_1 + u_2 \to B_2^1 = \left[B_{\bar{1}}^0, B_2^0\right] = D_{10}^{\overline{11}} + D_{12}^{\bar{9}} - D_{13}^{\bar{7}}\,. \tag{31}$$

Note that the root corresponding to the highest level, equation (31), is precisely the negative of α_0, where α_0 is the extra root added to the Dynkin diagram of F_4 to produce the extended Dynkin diagram.

In addition to the above 24 e-type generators, equations (16)–(31), I have 24 f-type generators obtained by taking the Hermitian conjugate of the above. Thus corresponding to the expressions above for the simple (level zero) lowering generators e_i I have

$$\begin{aligned}
f_1 &= e_1^\dagger = B_2^{3\dagger} = B_3^2 = D_5^7 + D_6^9 + D_8^{10} \\
f_2 &= e_2^\dagger = B_3^{4\dagger} = B_4^3 = D_3^5 + D_4^6 + D_{10}^{11} \\
f_3 &= e_3^\dagger = B_4^{0\dagger} = B_0^4 = 2^{1/2}\left(D_{\bar{1}}^3 + D_1^3 + D_2^4 + D_6^8 + D_9^{10} + D_{11}^{12}\right) \\
f_4 &= e_4^\dagger = S^{++++\dagger} = S^{----} = 2^{1/2}\left(z^* D_{\bar{1}}^2 + z D_1^2 + D_3^4 + D_5^6 + D_7^9 + D_{12}^{13}\right)
\end{aligned} \tag{32}$$

and so on for the generators in higher levels.

Moreover, for the Hermitian Cartan generators I have that the Chevalley and B_4 bases are related as follows:

$$\begin{aligned}
h_1 &= \left[f_1, e_1\right] = \left[B_3^2, B_2^3\right] = B_3^3 - B_2^2 \\
h_2 &= \left[f_2, e_2\right] = \left[B_4^3, B_3^4\right] = B_4^4 - B_3^3 \\
h_3 &= \left[f_3, e_3\right] = \left[B_0^4, B_4^0\right] = -B_4^4 \\
h_4 &= \left[f_4, e_4\right] = \left[S^{----}, S^{++++}\right] = \tfrac{1}{2}\left(B_1^1 + B_2^2 + B_3^3 + B_4^4\right)
\end{aligned} \tag{33}$$

or, solving above for the B_α^α and using (10)–(13),

$$\begin{aligned}
-B_1^1 &= h_1 + 2h_2 + 3h_3 + 2h_4 \\
&\qquad = \tfrac{1}{2}\left(D_2^2 + D_4^4 + D_6^6 + D_8^8 + D_9^9 + D_{10}^{10} + D_{11}^{11} + D_{12}^{12} + 2D_{13}^{13}\right) \\
-B_2^2 &= h_1 + h_2 + h_3 = \tfrac{1}{2}\left(D_2^2 + D_4^4 + D_6^6 - 2D_7^7 + D_8^8 - D_9^9 - D_{10}^{10} - D_{11}^{11} - D_{12}^{12}\right) \\
-B_3^3 &= h_2 + h_3 = \tfrac{1}{2}\left(D_2^2 + D_4^4 - 2D_5^5 - D_6^6 - D_8^8 + D_9^9 + D_{10}^{10} - D_{11}^{11} - D_{12}^{12}\right) \\
-B_4^4 &= h_3 = \tfrac{1}{2}\left(D_2^2 - 2D_3^3 - D_4^4 + D_6^6 - D_8^8 + D_9^9 - D_{10}^{10} + D_{11}^{11} - D_{12}^{12}\right).
\end{aligned} \tag{34}$$

4. The projected generators $\tilde{D}_a^b$

Now since the 26 is the defining representation of D_{13}, the results, above expressing the generators of F_4 in the 26-dimensional representation in terms of the generators of D_{13} in the 26-dimensional representation, can be interpreted as giving the generators of F_4 in terms of those of D_{13} in any representation. Then the $\tilde{D}_a^b$, the generators of D_{13} projected into F_4, are given by inverting the above equations.

Thus the 13 Cartan generators of D_{13} projected into F_4 are given by inverting (34):

$$\begin{aligned}
&\tilde{D}_1^1 = 0 \qquad \tilde{D}_2^2 = -\tfrac{1}{6}\left(B_1^1 + B_2^2 + B_3^3 + B_4^4\right) \qquad \tilde{D}_3^3 = \tfrac{1}{3}B_4^4 \\
&\tilde{D}_4^4 = -\tfrac{1}{6}\left(B_1^1 + B_2^2 + B_3^3 - B_4^4\right) \qquad \tilde{D}_5^5 = \tfrac{1}{3}B_3^3 \\
&\tilde{D}_6^6 = -\tfrac{1}{6}\left(B_1^1 + B_2^2 - B_3^3 + B_4^4\right) \qquad \tilde{D}_7^7 = \tfrac{1}{3}B_2^2 \\
&\tilde{D}_8^8 = -\tfrac{1}{6}\left(B_1^1 + B_2^2 - B_3^3 - B_4^4\right) \qquad \tilde{D}_9^9 = -\tfrac{1}{6}\left(B_1^1 - B_2^2 + B_3^3 + B_4^4\right) \\
&\tilde{D}_{10}^{10} = -\tfrac{1}{6}\left(B_1^1 - B_2^2 + B_3^3 - B_4^4\right) \qquad \tilde{D}_{11}^{11} = -\tfrac{1}{6}\left(B_1^1 - B_2^2 - B_3^3 + B_4^4\right) \\
&\tilde{D}_{12}^{12} = -\tfrac{1}{6}\left(B_1^1 - B_2^2 - B_3^3 - B_4^4\right) \qquad \tilde{D}_{13}^{13} = -\tfrac{1}{3}B_1^1\,.
\end{aligned} \tag{35}$$

Perhaps an explanation of how (35) is obtained is in order. Equations (34) are four equations for four B_α^α in terms of thirteen D_a^a (no summations). In addition there are nine more equations for appropriate components of the **273**-plet involving these same thirteen D_a^a. This total of 13 equations can be written as follows:

$$b_A = U_{AB}d_B \tag{36}$$

where $1 \leqslant A, B \leqslant 13$, where $d_B \equiv D_B^B$ (no summation), $b_A \equiv B_A^A$ (no summation) for $A = 1, 2, 3, 4$, and b_A for $5 \leqslant A \leqslant 13$ refers to components of the **273**-plet. Inversion of (36) is achieved by

$$d_A = U_{AB}^{-1}b_B \tag{37}$$

where the inverse of the 13×13 matrix U is given by

$$U^{-1} = \tfrac{1}{3}U^{\dagger} \tag{38}$$

where the factor $\frac{1}{3}$ accounts for the difference in the normalization of the d_A and b_A. Finally the projected $\tilde{d}_A$ are obtained by setting in (37) $b_A = 0$ for $5 \leqslant A \leqslant 13$.

By proceeding in the same fashion I obtain the 156 generators $\tilde{D}_a^b$ with $a > b$ by inverting the 24 e-type equations with the result:
for the 24 $\tilde{D}_{13}^b$ with $13 > b$:

$$\begin{aligned}
&\tilde{D}_{13}^{\overline{12}} = \tilde{D}_{13}^{\overline{11}} = \tilde{D}_{13}^{\overline{10}} = \tilde{D}_{13}^{\bar{9}} = \tilde{D}_{13}^{\bar{8}} = \tilde{D}_{13}^{\bar{6}} = \tilde{D}_{13}^{\bar{4}} = \tilde{D}_{13}^{\bar{2}} = 0 \\
&\tilde{D}_{13}^{\bar{7}} = -\tfrac{1}{3}B_2^1 \qquad \tilde{D}_{13}^{\bar{5}} = -\tfrac{1}{3}B_3^1 \qquad \tilde{D}_{13}^{\bar{3}} = -\tfrac{1}{3}B_4^1 \\
&\tilde{D}_{13}^{\bar{1}} = \tilde{D}_{13}^{1} = -\frac{1}{3\sqrt{2}}B_0^1 \qquad \tilde{D}_{13}^{2} = -\frac{1}{3\sqrt{2}}S^{+---} \\
&\tilde{D}_{13}^{3} = \tfrac{1}{3}B_{\bar{1}}^4 \qquad \tilde{D}_{13}^{4} = -\frac{1}{3\sqrt{2}}S^{+--+} \qquad \tilde{D}_{13}^{5} = \tfrac{1}{3}B_{\bar{1}}^3 \\
&\tilde{D}_{13}^{6} = -\frac{1}{3\sqrt{2}}S^{+-+-} \qquad \tilde{D}_{13}^{7} = \tfrac{1}{3}B_{\bar{1}}^2 \qquad \tilde{D}_{13}^{8} = -\frac{1}{3\sqrt{2}}S^{+-++} \\
&\tilde{D}_{13}^{9} = \frac{1}{3\sqrt{2}}S^{++--} \qquad \tilde{D}_{13}^{10} = \frac{1}{3\sqrt{2}}S^{++-+} \\
&\tilde{D}_{13}^{11} = -\frac{1}{3\sqrt{2}}S^{+++-} \qquad \tilde{D}_{13}^{12} = \frac{1}{3\sqrt{2}}S^{++++}
\end{aligned} \tag{39}$$

for the 22 $\tilde{D}^b_{12}$ with $12 > |b|$:

$$\begin{aligned}
&\tilde{D}^{\overline{11}}_{12} = \tilde{D}^{\overline{10}}_{12} = \tilde{D}^{\bar{8}}_{12} = \tilde{D}^{\bar{7}}_{12} = \tilde{D}^{\bar{5}}_{12} = \tilde{D}^{\bar{3}}_{12} = \tilde{D}^{2}_{12} = 0 \\
&\tilde{D}^{\bar{9}}_{12} = \tfrac{1}{3} B^1_2 \qquad \tilde{D}^{\bar{6}}_{12} = \tfrac{1}{3} B^1_3 \qquad \tilde{D}^{\bar{4}}_{12} = \tfrac{1}{3} B^1_4 \\
&\tilde{D}^{\bar{2}}_{12} = \frac{1}{3\sqrt{2}} B^1_0 \qquad \tilde{D}^{\bar{1}}_{12} = \frac{z}{3\sqrt{2}} S^{+---} \qquad \tilde{D}^{1}_{12} = \frac{z^*}{3\sqrt{2}} S^{+---} \\
&\tilde{D}^{3}_{12} = \frac{1}{3\sqrt{2}} S^{+--+} \qquad \tilde{D}^{4}_{12} = \tfrac{1}{3} B^{\bar{3}}_2 \qquad \tilde{D}^{5}_{12} = \frac{1}{3\sqrt{2}} S^{+-+-} \\
&\tilde{D}^{6}_{12} = \tfrac{1}{3} B^{\bar{2}}_4 \qquad \tilde{D}^{7}_{12} = -\frac{1}{3\sqrt{2}} S^{++--} \qquad \tilde{D}^{8}_{12} = \frac{1}{3\sqrt{2}} B^0_2 \\
&\tilde{D}^{9}_{12} = \tfrac{1}{3} B^{\bar{4}}_3 \qquad \tilde{D}^{10}_{12} = \frac{1}{3\sqrt{2}} B^{\bar{3}}_0 \qquad \tilde{D}^{11}_{12} = \frac{1}{3\sqrt{2}} B^0_4
\end{aligned} \tag{40}$$

for the 20 $\tilde{D}^b_{11}$ with $11 > |b|$:

$$\begin{aligned}
&\tilde{D}^{\bar{9}}_{11} = \tilde{D}^{\bar{7}}_{11} = \tilde{D}^{\bar{6}}_{11} = \tilde{D}^{\bar{5}}_{11} = \tilde{D}^{3}_{11} = \tilde{D}^{4}_{11} = 0 \\
&\tilde{D}^{\overline{10}}_{11} = -\tfrac{1}{3} B^1_2 \qquad \tilde{D}^{\bar{8}}_{11} = -\tfrac{1}{3} B^1_3 \qquad \tilde{D}^{\bar{4}}_{11} = \frac{1}{3\sqrt{2}} B^1_0 \\
&\tilde{D}^{\bar{3}}_{11} = -\frac{1}{3\sqrt{2}} S^{+---} \qquad \tilde{D}^{\bar{2}}_{11} = \tfrac{1}{3} B^4_{\bar{1}} \qquad \tilde{D}^{\bar{1}}_{11} = \frac{z^*}{3\sqrt{2}} S^{+--+} \\
&\tilde{D}^{1}_{11} = \frac{z}{3\sqrt{2}} S^{+--+} \qquad \tilde{D}^{2}_{11} = \tfrac{1}{3} B^{\bar{3}}_2 \qquad \tilde{D}^{5}_{11} = -\frac{1}{3\sqrt{2}} S^{+-++} \\
&\tilde{D}^{6}_{11} = \frac{1}{3\sqrt{2}} B^{\bar{2}}_0 \qquad \tilde{D}^{7}_{11} = \frac{1}{3\sqrt{2}} S^{++-+} \qquad \tilde{D}^{8}_{11} = -\tfrac{1}{3} B^4_2 \\
&\tilde{D}^{9}_{11} = \frac{1}{3\sqrt{2}} B^0_2 \qquad \tilde{D}^{10}_{11} = \tfrac{1}{3} B^4_3
\end{aligned} \tag{41}$$

for the 18 $\tilde{D}^b_{10}$ with $10 > |b|$:

$$\begin{aligned}
&\tilde{D}^{\bar{9}}_{10} = \tilde{D}^{\bar{7}}_{10} = \tilde{D}^{\bar{4}}_{10} = \tilde{D}^{\bar{3}}_{10} = \tilde{D}^{5}_{10} = \tilde{D}^{6}_{10} = 0 \\
&\tilde{D}^{\bar{8}}_{10} = -\tfrac{1}{3} B^1_4 \qquad \tilde{D}^{\bar{5}}_{10} = \frac{1}{3\sqrt{2}} S^{+---} \qquad \tilde{D}^{\bar{2}}_{10} = -\tfrac{1}{3} B^3_{\bar{1}} \\
&\tilde{D}^{\bar{1}}_{10} = -\frac{z^*}{3\sqrt{2}} S^{+-+-} \qquad \tilde{D}^{1}_{10} = -\frac{z}{3\sqrt{2}} S^{+-+-} \qquad \tilde{D}^{2}_{10} = \tfrac{1}{3} B^{\bar{4}}_2 \\
&\tilde{D}^{3}_{10} = -\frac{1}{3\sqrt{2}} S^{+-++} \qquad \tilde{D}^{4}_{10} = \frac{1}{3\sqrt{2}} B^{\bar{2}}_0 \qquad \tilde{D}^{7}_{10} = \frac{1}{3\sqrt{2}} S^{+++-} \\
&\tilde{D}^{8}_{10} = \tfrac{1}{3} B^3_2 \qquad \tilde{D}^{9}_{10} = \frac{1}{3\sqrt{2}} B^0_4
\end{aligned} \tag{42}$$

for the 16 $\tilde{D}^b_9$ with $9 > |b|$:

$$\tilde{D}^{\bar{7}}_9 = \tilde{D}^{\bar{2}}_9 = \tilde{D}^{3}_9 = \tilde{D}^{5}_9 = \tilde{D}^{8}_9 = 0$$
$$\tilde{D}^{\bar{8}}_9 = -\frac{1}{3\sqrt{2}} B^1_0 \qquad \tilde{D}^{\bar{6}}_9 = \tfrac{1}{3} B^1_4 \qquad \tilde{D}^{\bar{5}}_9 = -\frac{1}{3\sqrt{2}} S^{+--+}$$
$$\tilde{D}^{\bar{4}}_9 = \tfrac{1}{3} B^3_{\bar{1}} \qquad \tilde{D}^{\bar{3}}_9 = \frac{1}{3\sqrt{2}} S^{+-+-} \qquad \tilde{D}^{\bar{1}}_9 = -\frac{z}{3\sqrt{2}} S^{+-++} \tag{43}$$
$$\tilde{D}^{1}_9 = -\frac{z^*}{3\sqrt{2}} S^{+-++} \qquad \tilde{D}^{2}_9 = \frac{1}{3\sqrt{2}} B^0_2 \qquad \tilde{D}^{4}_9 = \tfrac{1}{3} B^4_2$$
$$\tilde{D}^{6}_9 = \tfrac{1}{3} B^3_2 \qquad \tilde{D}^{7}_9 = \frac{1}{3\sqrt{2}} S^{++++}$$

for the 14 $\tilde{D}^b_8$ with $8 > |b|$:

$$\tilde{D}^{\bar{6}}_8 = \tilde{D}^{\bar{5}}_8 = \tilde{D}^{\bar{4}}_8 = \tilde{D}^{\bar{3}}_8 = \tilde{D}^{7}_8 = 0$$
$$\tilde{D}^{\bar{7}}_8 = -\frac{1}{3\sqrt{2}} S^{+---} \qquad \tilde{D}^{\bar{2}}_8 = \tfrac{1}{3} B^2_{\bar{1}} \qquad \tilde{D}^{\bar{1}}_8 = -\frac{z^*}{3\sqrt{2}} S^{++--}$$
$$\tilde{D}^{1}_8 = -\frac{z}{3\sqrt{2}} S^{++--} \qquad \tilde{D}^{2}_8 = \tfrac{1}{3} B^{\bar{4}}_3 \qquad \tilde{D}^{3}_8 = -\frac{1}{3\sqrt{2}} S^{++-+} \tag{44}$$
$$\tilde{D}^{4}_8 = -\frac{1}{3\sqrt{2}} B^0_3 \qquad \tilde{D}^{5}_8 = \frac{1}{3\sqrt{2}} S^{+++-} \qquad \tilde{D}^{6}_8 = \frac{1}{3\sqrt{2}} B^0_4$$

for the 12 $\tilde{D}^b_7$ with $7 > |b|$:

$$\tilde{D}^{2}_7 = \tilde{D}^{4}_7 = \tilde{D}^{6}_7 = 0 \qquad \tilde{D}^{\bar{6}}_7 = -\frac{1}{3\sqrt{2}} S^{+--+} \qquad \tilde{D}^{\bar{5}}_7 = \tfrac{1}{3} B^{\bar{3}}_2$$
$$\tilde{D}^{\bar{4}}_7 = \frac{1}{3\sqrt{2}} S^{+-+-} \qquad \tilde{D}^{\bar{3}}_7 = \tfrac{1}{3} B^{\bar{4}}_2 \qquad \tilde{D}^{\bar{2}}_7 = -\frac{1}{3\sqrt{2}} S^{+-++} \tag{45}$$
$$\tilde{D}^{\bar{1}}_7 = \tilde{D}^{1}_7 = \frac{1}{3\sqrt{2}} B^0_2 \qquad \tilde{D}^{3}_7 = \tfrac{1}{3} B^4_2 \qquad \tilde{D}^{5}_7 = \tfrac{1}{3} B^3_2$$

for the 10 $\tilde{D}^b_6$ with $6 > |b|$:

$$\tilde{D}^{\bar{5}}_6 = \tilde{D}^{\bar{2}}_6 = \tilde{D}^{3}_6 = 0 \qquad \tilde{D}^{\bar{4}}_6 = -\tfrac{1}{3} B^2_{\bar{1}} \qquad \tilde{D}^{\bar{3}}_6 = \frac{1}{3\sqrt{2}} S^{++--}$$
$$\tilde{D}^{\bar{1}}_6 = -\frac{z}{3\sqrt{2}} S^{++-+} \qquad \tilde{D}^{1}_6 = -\frac{z^*}{3\sqrt{2}} S^{++-+} \qquad \tilde{D}^{2}_6 = \frac{1}{3\sqrt{2}} B^0_3 \tag{46}$$
$$\tilde{D}^{4}_6 = \tfrac{1}{3} B^4_3 \qquad \tilde{D}^{5}_6 = \frac{1}{3\sqrt{2}} S^{++++}$$

for the eight $\tilde{D}^b_5$ with $5 > |b|$:

$$\tilde{D}^{2}_5 = \tilde{D}^{4}_5 = 0 \qquad \tilde{D}^{\bar{4}}_5 = \frac{1}{3\sqrt{2}} S^{++--} \qquad \tilde{D}^{\bar{3}}_5 = \tfrac{1}{3} B^{\bar{4}}_3$$
$$\tilde{D}^{\bar{2}}_5 = -\frac{1}{3\sqrt{2}} S^{++-+} \qquad \tilde{D}^{\bar{1}}_5 = \tilde{D}^{1}_5 = \frac{1}{3\sqrt{2}} B^0_3 \qquad \tilde{D}^{3}_5 = \tfrac{1}{3} B^4_3 \tag{47}$$

for the six D_4^b with $4 > |b|$:

$$\tilde{D}_4^{\bar{3}} = \tilde{D}_4^{\bar{2}} = 0 \qquad \tilde{D}_4^{\bar{1}} = -\frac{z}{3\sqrt{2}}S^{+++-} \qquad \tilde{D}_4^{1} = -\frac{z^*}{3\sqrt{2}}S^{+++-}$$
$$\tilde{D}_4^{2} = \frac{1}{3\sqrt{2}}B_4^0 \qquad \tilde{D}_4^{3} = \frac{1}{3\sqrt{2}}S^{++++} \tag{48}$$

for the four $\tilde{D}_3^b$ with $3 > |b|$:

$$\tilde{D}_3^{2} = 0 \qquad \tilde{D}_3^{\bar{2}} = -\frac{1}{3\sqrt{2}}S^{+++-} \qquad \tilde{D}_3^{\bar{1}} = \tilde{D}_3^{1} = \frac{1}{3\sqrt{2}}B_4^0 \tag{49}$$

and finally for the two $\tilde{D}_2^b$ with $2 > |b|$:

$$\tilde{D}_2^{\bar{1}} = \frac{z^*}{3\sqrt{2}}S^{++++} \qquad \tilde{D}_2^{1} = \frac{z}{3\sqrt{2}}S^{++++}\,. \tag{50}$$

Lastly, the 156 $\tilde{D}_a^b$ with $a < b$ are obtained from the results above by Hermitian conjugation:

$$\tilde{D}_b^a = \tilde{D}_a^{b\dagger} \qquad B_b^a = B_a^{b\dagger} \qquad S^{\overline{pqrs}} = S^{pqrs\dagger}\,. \tag{51}$$

This completes the calcultion of the Casimir operators of F_4.

5. Conclusions

I conclude with two remarks.

(i) For $k = 2$ the result of inserting the explicit formulae for the projected $\tilde{D}$, (35), (39)–(51), into (6) can be simplified into the following formula for the quadratic Casimir operator of F_4:

$$\mathcal{C}_2(F_4) = \tilde{D}_a^b \tilde{D}_b^a = \tfrac{1}{3} B_\alpha^\beta B_\beta^\alpha + \tfrac{2}{3} S^{pqrs} S^{\overline{pqrs}} \tag{52}$$

and I remind the reader that the various subscripts are summed over the following range: $-13 \leqslant a, b \leqslant 13$ (zero excluded); $-4 \leqslant \alpha, \beta \leqslant 4$ (zero included); $p, q, r, s = \pm$.

The general form of this result for the quadratic Casimir of F_4 in the B_4 basis was to be expected since the two pieces in (52) are the only quadratic invariants of the subgroup B_4 that can be formed out of the adjoint **36** and the spinor **16** of B_4. Thus this result can be viewed as a test of the formalism.

(ii) Recall that, according to (4), the independent Casimirs are of degree $k = 2s$, $1 \leqslant s \leqslant 13$. Now consider the *Cartan* part of the Casimirs. If I denote the Cartan part of $\mathcal{C}_k(F_4)$ by $\mathcal{K}_k$ then it follows from (6) that

$$\mathcal{K}_k = \left(\tilde{D}_a^a\right)^k. \tag{53}$$

Since $\tilde{D}_{\bar{a}}^{\bar{a}} = -\tilde{D}_a^a$ this is manifestly zero for $k =$ odd. For $k =$ even, equation (53) becomes (where I have set $b_\alpha \equiv B_\alpha^\alpha$, no summation)

$$\begin{aligned}\mathcal{K}_k = 2\sum_{a=1}^{13}\left(\tilde{D}_a^a\right)^k = 2\times 6^{-k}\big\{&(b_1+b_2+b_3+b_4)^k + (2b_4)^k + (b_1+b_2+b_3-b_4)^k \\ &+(2b_3)^k + (b_1+b_2-b_3+b_4)^k + (2b_2)^k + (b_1+b_2-b_3-b_4)^k \\ &+(b_1-b_2+b_3+b_4)^k + (b_1-b_2+b_3-b_4)^k + (b_1-b_2-b_3+b_4)^k \\ &+(b_1-b_2-b_3-b_4)^k + (2b_1)^k\big\}.\end{aligned} \tag{54}$$

For $k = 2$, equation (54) gives

$$\mathcal{K}_2 = \tfrac{2}{3}\left(b_2^2 + b_2^2 + b_3^2 + b_4^2\right) \tag{55}$$

while for $k = 4$ it gives

$$\mathcal{K}_4 = 3^{-3}\left(b_1^2 + b_2^2 + b_3^2 + b_4^2\right)^2 . \tag{56}$$

This proves that the degree-4 Casimir is not functionally independent of the degree-2 Casimir.

For $k = 6$, equation (54) gives

$$\begin{aligned}\mathcal{K}_6 = 2^{-2} \times 3^{-5}\{&3[b_1^6 + b_2^6 + b_3^6 + b_4^6] + 5[b_1^4(b_2^2 + b_3^2 + b_4^2) + b_2^4(b_1^2 + b_3^2 + b_4^2)\\ &+ b_3^4(b_1^2 + b_2^2 + b_4^2) + b_4^4(b_1^2 + b_2^2 + b_3^2)]\\ &+ 30[b_1^2(b_2^2 b_3^2 + b_3^2 b_4^2 + b_2^2 b_4^2) + b_2^2 b_3^2 b_4^2]\}\end{aligned} \tag{57}$$

which *is* functionally independent of the degree-2 Casimir (were it proportional to the cube of the degree-2 Casimir it would have the coefficient of the expression in the first, second and third square bracket in the ratio 1:3:6 instead of the 3:5:30 above).

Continuing along these lines I find that the degree 8 is functionally independent of the degree 2 and 6, while the degree 10 is dependent:

$$\mathcal{K}_{10} \sim \mathcal{K}_2\{28\mathcal{K}_2(\mathcal{K}_2^3 - \mathcal{K}_6) + 3\mathcal{K}_8\} \tag{58}$$

and lastly the degree 12 is independent of those of lower degree. Since all the Casimirs are functions of the four quantities b_α^2, $1 \leqslant \alpha \leqslant 4$, I can solve for the b_α^2 in terms of the independent Casimirs of degree 2, 6, 8 and 12, and consequently all Casimirs of higher degree are necessarily dependent. This completes the demonstration that the independent Casimirs are those of degree equal to the exponents of F_4 plus one.

Acknowledgment

Many useful discussions with Charlie Goebel are gratefully acknowledged.

References

[1] Berdjis F 1981 *J. Math. Phys.* **22** 1851
Berdjis F and Beslmuller E 1981 *J. Math. Phys.* **22** 1857
[2] Hughes J W B and Van der Jeugt J 1985 *J. Math. Phys.* **26** 894
[3] Bincer A M and Riesselmann K 1993 *J. Math. Phys.* **34** 5935
[4] Bincer A M *Can. J. Phys.* to be published
[5] Perelomov A M and Popov V S 1968 *Sov. J. Nucl. Phys.* **7** 290
[6] Patera J 1971 *J. Math. Phys.* **12** 384
[7] Ekins J M and Cornwell J F 1973 *Rep. Math. Phys.* **7** 167